A Survey of Predicativity

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1 The Early Years

1.1 Russell and Poincaré

The word 'predicative' first appeared in Russell's note *On Some Difficulties in the Theory of Transfinite Numbers and Order Types* [Rus07]. Paradoxes such as Russell's Paradox show that we cannot form the class $\{x \mid \phi(x)\}$ for all propositional functions $\phi(x)$. Russell proposed we call $\phi(x)$ *predicative* if it defines a class and *non-predicative* otherwise, but did not offer a criterion by which we could decide which propositional functions are which.

A first such criterion was offered by Poincaré in the third part of his paper *Les Mathématiques et la Logique* [Poi06]. He proposed the *vicious circle principle*: "The definitions which ought to be regarded as non-predicative are those which contain a vicious circle." (p. 1063) He indicated by example what he meant by 'vicious circle': in both the Richard paradox and the Burali-Forti paradox, we define an aggregate E, and make use of E within its own definition.

Poincaré proposed that definitions involving such a 'vicious circle' are illegitimate: "A definition containing a vicious circle defines nothing." [Poi06, p. 1065]

Poincaré justified this as follows. For a definition to be legitimate, it must be possible to substitute the definiens for the defined term. Recursive definitions are acceptable, as this substitution will always terminate. But in the case of a definition involving a vicious circle, the substitution will never terminate.

Russell agreed with Poincaré, and offered a more formal version of the vicious-circle principle in *Les Paradoxes de la Logique* [Rus06]:

> Whatever involves an apparent variable must not be among the possible values of that variable.

(An 'apparent variable' is a bound variable in modern terminology.) In a few years, he would offer a system of logic that adhered to this vicious-circle principle, the *theory of types*, first appearing in the paper *Mathematical Logic as Based on the Theory of Types* [Rus08], and later forming the basis of Whitehead and Russell's *Principia Mathematica* [WR27]. An historical account of the theory of types is presented in [KLN02].

1.2 Sets and Orders

The types in Russell's theory include the following. There is a type of *individuals*; and, given any types A and $B_1, \ldots, B_n$, we may form the type of sets $\{x \mid \phi(x)\}$, where x ranges over A and the quantifiers in $\phi(x)$ range over the types $B_1, \ldots, B_n$ only. This type is said to be *higher* than the types $B_1, \ldots, B_n$. Thus, in defining an object of a given type, one may only quantify over lower types. In particular, one may not quantify over a type C when defining an object of type C.

We give names to some of these types. The 0th-order sets of individuals, or *arithmetic* sets[1], are those that can be defined as $\{x \mid \phi(x)\}$, where all the quantifiers in $\phi(x)$ range over individuals only. The 1st-order sets are those that can be defined using quantifiers over numbers and quantifiers over 0th-order sets. The 2nd-order sets are those that can be defined using quantification over numbers, 0th-order sets and 1st-order sets; and so forth.[2]

The ramification of sets into orders has some undesirable consequences. If we construct the real numbers as sets of rationals, we find ourselves with many different sets of reals: the 0th-order reals, the 1st-order reals, and so forth. The completeness property does not hold for any of these sets of reals; the least upper bound of a bounded set of nth-order reals is, in general, a real of order $n + 1$. It is impractical to carry out real analysis in such a setting. Similar problems can be expected to arise in many other branches of mathematics.

Russell's solution, introduced in [Rus08] and used in *Pricipia Mathematica* [WR27] (with which, however, Whitehead and Russell were not satisfied) was to introduce the *Axiom of Reducibility.* With respect to the types introduced above, the axiom reads: for any nth-order set S, there exists a 0th-order set T with the same members as S. This axiom effectively allows one to perform impredicative definitions in the theory of types.

Weyl's solution to this same problem in *Das Kontinuum* [Wey18] was to remove all orders above the 0th; to see how much of mathematics we can reconstruct while restricting ourselves to the arithmetic sets. Many theorems of mathematics cannot be proved in this system, but often a weaker theorem can be proved which is almost as useful. For example, the completeness property of the reals does not hold in this system, but two weaker principles do hold: that every bounded set of *rationals* has a real least upper bound, and that every bounded *sequence* of reals has a least upper bound. Weyl showed that these principles were sufficient for many of the purposes for which the completeness property is used.

[1] Russell called these the *predicative* classes. We avoid this terminology, as it is a very different meaning of the word to the one we are considering in this article. Weyl called these the *finite* sets [Wey18].

[2] Strictly, the theory of types deals with types of propositional functions (predicates), not of sets. Statements involving the set $\{x \mid \phi(x)\}$ are seen as shorthand for statements involving the propositional function $\phi(x)$. The theory of types also included types of propositions, and of binary propositional functions, ternary propositional functions, and so forth.

2 Predicativity Modulo the Natural Numbers

2.1 Wang-Kreisel Predicativity

Instead of restricting ourselves to order 0, we can pursue the opposite approach and introduce transfinite orders. Gödel [G$\ddot{4}$4] suggested constructing the sets of order α, where α ranges the ordinals. For each successor ordinal $\alpha + 1$, the sets of order $\alpha + 1$ are those that can be defined by quantifying over the sets of order α. For each limit ordinal λ, the collection of sets of order λ is the union of the collections of sets of order α for $\alpha < \lambda$. This idea was later named the *ramified analytic hierarchy* by Feferman [Fef05]. Following Feferman's notation, let us denote by R_α the set of all sets of order α.

Gödel showed that we can prove impredicatively that there exists an ordinal α such that every set in the ramified hierarchy is coextensional with a set of order α. Thus, we could pursue classical mathematics, making use of sets of order α alone. However, the ordinal α cannot be defined predicatively.

Lorenzen [Lor55] (according to Feferman [Fef05]) and Wang [Wan54] suggested restricting the ramified hierarchy to the recursive ordinals. To formalize this idea, Wang introduced a sequence of formal systems Σ_α, where α ranges over the recursive ordinals. The intended interpretation of Σ_α comprises the natural numbers and the sets of order $\leq \alpha$.

If we extend the hierarchy over all the recursive ordinals, Kleene [Kle59] showed that the sets obtained are exactly the *hyperarithmetic* sets; that is, those sets definable by both a Π^1_1-formula and a Σ^1_1-formula. Spector [Spe55] showed that the hyperarithmetic well-orderings are exactly the recursive well-orderings. In 1960, Kreisel [Kre60a] argued that these and other results suggest that the predicatively definable sets should be identified with the hyperarithmetic sets.

We may argue for this conclusion as follows[3]. Consider a predicativist, who may only construct a set if it is defined using a definiton that involves only previously constructed sets. If he has constructed a well-ordering of order type α, he may use transifinite recursion over this well-ordering to define the set R_α, and so can construct all the sets of order $\leq \alpha$. The predicativist begins with the arithmetic sets. For any recursive ordinal α, there is an arithmetic well-ordering of order type α. Thus, the predicativist may construct all the sets R_α for α recursive, and hence all the hyperarithmetic sets. These sets do not include any well-orderings with non-recursive ordinal, and so the procedure comes to a halt. The subsets of $\mathbb{N}$ the predicativist has been able to construct are exactly the hyperarithmetic sets.

The following is the important premise of this argument. For reference, we shall refer to it as *Wang-Kreisel's principle*:

> If a well-ordering W of order type α can be predicatively defined, then the sets of order $\leq \alpha$ are predicatively definable.

[3]This argument is not Kreisel's. It is similar to the argument given in Feferman [Fef05].

2.2 Feferman-Schütte Predicativity

A few years before his 1960 paper, Kreisel [Kre60b] suggested modifying Wang's heirarchy as follows. It is not enough that the well-ordering W can be *defined* predicatively; the predicativist must also be able to *prove* that W is a well-ordering. Only then is the system Σ_α predicatively acceptable. However, he wrote "I have no information about the least ordinal not obtained by these extensions starting from Z."

While Kreisel had abandoned this proposal by 1960 `Check this`, it was taken up independently in 1964–5 by Feferman [Fef64] and Schütte [Sch65], who showed that the ordinals obtained by this process are exactly those below Γ_0, the least ordinal such that $\chi_{\Gamma_0}(0) = \Gamma_0$, where $\chi_\alpha(\beta)$ is the Veblen critical function.

We record the new principle as *Feferman-Schütte's principle*:

> If a well-ordering W of order type α can be predicatively defined and predicatively proven to be a well-ordering, then the sets of order $\leq \alpha$ are predicatively definable.

Given a hierarchy of systems S_α, Feferman defined the *autonomous* ordinals with respect to this hierarchy to be the least initial segment of the ordinals that contains ω and is closed under addition and the clause:

> If α is autonomous and S_α proves that there exists a well-ordering of length β, then β is autonomous.

In [Fef64], Feferman gave two hierarchies of systems intended to characterise predicative reasoning and showed that, for either hierarchy, the autonomous ordinals are exactly those below Γ_0. Schütte independently gave another, with the same conclusion. `Check the differences between Wang's, Kreisel's, Feferman's and Schutte's systems.` This was held to be strong justification for identifying the predicatively definable sets with the sets of order $< \Gamma_0$. One consequence of this is that the systems with proof-theoretic ordinal $< \Gamma_0$ are exactly those systems that can be predicatively proven to be consistent.

2.3 Weaver's Objection

Since the 1960s, the logical community has generally accepted Feferman and Schütte's criterion as the definition of predicativity. Recently, however, Weaver [Wea05] has argued that much larger ordinals that Γ_0 should be considered predicatively acceptable. He has raised two objections to Feferman's argument above.

Firstly, the argument above uses the principle that, if we have proved W to be a well-order, then we may perform transfinite induction over W. This inference is justified classically, but not necessarily predicatively. In particular, proving W is a well-order allows one to conclude

$$\forall x(\forall y(\langle y, x\rangle \in W \to y \in X) \to x \in X) \to \forall x x \in X$$

for any *set* X, but not

$$\forall x(\forall y(\langle y, x\rangle \in W \to \phi(y)) \to \phi(x)) \to \forall x\phi(x)$$

for any *formula* $\phi(x)$, which is what is required to construct R_α.

Weaver thus proposes replacing Feferman-Schütte's principle with the following, which we shall call *Weaver's principle*:

> If a well-ordering W of order type α can be predicatively defined and we can predicatively prove all formulas of the following schema:
>
> $$\forall x(\forall y(\langle y, x\rangle \in W \to \phi(y)) \to \phi(x)) \to \forall x\phi(x)$$
>
> then the sets of order $\leq \alpha$ are predicatively definable.

By itself, this modification would lead to a smaller class of ordinals being predicatively acceptable. However, Weaver also suggests a second principle

Let us say that a formal system is *predicatively acceptable* if all its theorems are predicatively provable. Then Feferman-Schütte's principle can be rewritten as

> If Σ_α is predicatively acceptable and $\Sigma_\alpha \vdash I(\beta)$, then Σ_β is predicatively acceptable.

or as

> (A) If $I(\beta)$ is predicatively provable and $\Sigma_\beta \vdash \phi$, then ϕ is predicatively provable.

Weaver suggests that the same principles that justify this reasoning also justify the schema

> (B) For any formula ϕ, the formula $I(x) \to P(x, \ulcorner\phi\urcorner) \to \phi$ is predicatively provable.

He shows that adding this schema to Σ_0 allows us to prove well-orderings of order type Γ_0 and higher exist.

Weaver also presents his own hierarchy of systems. He considers the operation of extending a formal system with a *truth predicate*; for any system S, the theory $Tarski(S)$ extends S with a predicate Tx, intended to express "x is the Gödel number of a formula that is true in every model of S." The extra axioms of $Tarski(S)$ express the conditions: the axioms of S are true; the rules of deduction of S preserve truth; if $\phi(\overline{n})$ is true for every natural number n then $\forall x\phi(x)$ is true.

He defines a way of transifinetly iterating the operation '$Tarski$', and proposes: if a predicativist accepts the system S, and S proves that transifinite recursion over α is possible, then the predicativist should accept the result of applying $Tarski$ to S α times. `Check the details here.` He presents a system that is predicatively acceptable by this principle, which can prove the existence of ordinals much larger than Γ_0. The upper limit on the ordinals that

can be proven to exist in this system is at least $\phi_{\Omega^\omega}(0)$; the precise upper limit is unknown.

Against this, Friedman ('Weaver's error?' on the FOM mailing list, 10 Apr 2006) has argued that, given (A), we need not accept (B). There is a difference between claiming 'If a predicativist accepts $I(\beta)$, and ϕ is a theorem of Σ_β, then there is a way for the predicativist to come to accept ϕ' and claiming 'A predicativist will accept the implication: if $I(\beta)$ and $\Sigma_\beta \vdash \phi$ then ϕ'. Feferman claims the first, but not the second.

Weaver replied ('Friedman's confusion?' on the FOM mailing list, 11 Apr 2006) that he accepts that (A) does not *entail* (B); nevertheless, it seems 'implausible under any conception of predicativism' that every instance of (A) should be true and (B) not true. If we can see by some means that (A) is true, a predicativist should be able to do the same; and if a predicativist can accept (A), then (B) is true. Feferman (`http://www.math.wustl.edu/~nweaver/response1.html`) replied (to a more general form of this argument), saying that it is possible that a predicativist might accept each instance of (A) by a different reasoning procedure, not by a uniform method that would lead to him accepting the implication in (B). Weaver (`http://www.math.wustl.edu/~nweaver/response2.html`) replied that the means by which we come to recognise (A) should also be available to the predicativist.

The argument raised with each side claiming the burden of proof lay with the other: Weaver's critics insisting that (A) does not entail (B), Weaver demanding a form of justification for (A) that does not also justify (B).

3 Inductive Definitions

Alongside the above historical development of the concept of predicativity, there has been a debate on whether inductive definitions should be considered as predicative definitions. Consider the inductive definition of the set of natural numbers:

1. 0 is a natural number.
2. The successor of any natural number is a natural number.
3. The natural numbers are exactly those objects that can be constructed by the first two clauses.

In his theory of types, Russell did not allow himself to introduce the natural numbers by an inductive definition, nor to take them as given; instead, he defined them. Having defined the concepts of cardinal number, zero and successor, he defined 'n is a natural number' to mean

for all sets X, if $0 \in X$ and $\forall x(x \in X \to x + 1 \in X)$, then $n \in X$.

As it stands, this is an impredicative definition. If we recast it so X only ranges over sets of order α for some α, then we will have 0th-order natural numbers,

1st-order natural numbers, and so forth; a similar problem to the one we faced with real numbers in Section 1.2 above. Whitehead and Russell's solution was to use the Axiom of Reducibility.

Wang [Wan59] agreed that, when an inductive definition is converted into an explicit definition by this method, the result is an impredicative definition. However, he argued that an inductive definition can be read as an 'implicit definition', in which case it is unobjectionable. He introduced a set of axioms for the predicative sets, which included:

(7) An inductive definition containing explicitly only predicative notions defines a predicative set.

(8) Given any predicative ordinal, the result of iterating an inductive definition scheme as many times as the ordinal yields a predicative set.

Wang states that, as a consequence of adopting these principles, the predicative sets of natural numbers are exactly the hyperarithmetic sets.

Lorenzen and Myhill [LM59] also argued that inductive definitions are acceptable to the predicativist, because they do not make use of the actual infinite.

Dummett [Dum63] disagreed, arguing that

> [T]he notion of 'natural number', even as characterised by the formal system, is impredicative. The totality of natural numbers is characterised as one for which induction is valid with respect to any well-defined property, where by a 'well-defined property' is understood one which is well defined relative to the totality of natural numbers.

Thus, however we interpret the inductive definition of $\mathbb{N}$, the interpretation must involve the validity of proving by induction a statement that involves quantification over $\mathbb{N}$, and this is an impredicativity.

Kreisel [Kre65, Kre60a] made the intriguing suggestion that inductive definitions may be impredicative when using classical logic, but may not be when using intuitionistic logic. He justified this by presenting an abstract theory of constructions, some of which are to be singled out as proofs. From the inductive definition of the natural numbers, we know what it means to prove that a given construction is a natural number. The validity of proof by induction rests in the fact that, whenever we are given a property P, proofs of $P(0)$ and $P(a) \to P(a+1)$, and a natural number a, we can construct a proof of $P(a)$. This last statement is justified by the theory of constructions, and not by reading the inductive definition as quantifying over all properties P.

More recently, Parsons [Par92] has argued that the word 'predicative' has been used with two different meanings historically. In the modern sense, sets are 'impredicatively defined if they are given by abstracts involving quantification over some totality of sets to which they themselves belong.' In this sense, inductive definitions are impredicative; Parsons' argument is very similar to Feferman's. However, Parsons argues Poincaré and Lorenzen and Myhill used the word in a weaker sense: for them, sets and classes should be viewed as the

extensions of predicates, and a definition is impredicative if it defines a set using a predicate that refers to the set itself. In this sense, inductive definitions are predicative, as Lorenzen and Myhill argued.

4 Predicativity in Type Theory

In general, the type theory community has confined its interest in predicativity to the question of whether a *universe* is predicative.

Roughly, a universe U is a type of types. We say U is *impredicative* if some of the types in U are constructed using U itself; for example, if $\Pi x : U.x$ has type U. Otherwise, U is *predicative*. A type theory is called *predicative* if it contains no impredicative universes.

Martin-Löf's original type theory [ML71] used a single universe V, which was highly impredicative as the system used the typing rule $V \in V$. After Girard [Gir72] showed that the system was inconsistent, Martin-Löf [ML75] modified the system so that the universe V is predicative. He called the objects in V the *small* types, and the other types the *large* types. This system has been referred to (e.g. by Aczel and Gambino [AG02]) as ML_1.

Later [ML82] Martin-Löf extended the system again to include *W-types*, types of well-ordered trees, and an infinite series of predicative universes V_0, V_1, V_2, ..., with $V_0 \in V_1 \in V_2 \in \cdots$ We shall refer to the systems with n universes, with and without W-types, as ML_n and ML_nW respectively, and the systems with infinitely many universes as ML and MLW.

Each of the types in ML is a simple inductive definition, and thus the system is predicative modulo the natural numbers in Feferman and Schütte's sense. It was proven independently by Feferman [Fef82] and Aczel[4] that the proof-theoretic strength of ML is Γ_0. Therefore, one can build a model of ML without using only constructions that are predicative modulo the natural numbers, and the ordinals that can be proven to exist in the system are exactly those less than Γ_0.

The definition of W-types is not a simple inductive definition, but a generalised inductive definition. Including W-types increases the strength of a type theory dramatically. Setzer [Set93] has shown that even ML_1W has a proof-theoretic ordinal that is much larger than Γ_0. If we agree with Parsons that there are two senses of the word 'predicative', then type theories with W-types are predicative in Lorenzen and Myhill's sense, but not in Feferman and Schütte's sense.

It is worth noting that consistent impredicative type theories have also been employed. For example, the Calculus of Constructions [CH88] uses a single impredicative universe, while ECC [Luo94] uses an impredicative universe below an infinite series of predicative universes.

[4]Palmgren [Pal98] claims without citation that Aczel made this discovery.

A Proof-Theoretic Ordinals of Type Theories

Ordinal Definitions The *Veblen functions* $\phi_\alpha(\beta)$ for all ordinals α, β are defined as follows:

- $\phi_0(\beta) = \omega^\beta$
- For $\alpha > 0$, the function ϕ_α enumerates the ordinals that are fixed points of ϕ_β for all $\beta < \alpha$.

Define the sequence of ordinals γ_n for n a natural number by

$$\begin{aligned}\gamma_0 &= \epsilon_0 \\ \gamma_{n+1} &= \phi_{\gamma_n}(0)\end{aligned}$$

Let $\Gamma_0 = \sup_{n\in\mathbb{N}} \gamma_n$.

Let I be the least weakly inaccessible cardinal (i.e. the least regular cardinal such that $\aleph_I = I$), and let I_n be the nth weakly inaccessible cardinal. For every regular cardinal $\kappa > \omega$, define the ordinal $\psi_\kappa(\alpha)$ and the set of ordinals $C(\alpha, \beta)$ simultaneously by transfinite recursion on α as follows:

- $\psi_\kappa(\alpha)$ is the least β such that $\kappa \in C(\alpha, \beta)$ and $C(\alpha, \beta) \cap \kappa \subseteq \beta$ (i.e. there is no $\lambda \in C(\alpha, \beta)$ such that $\beta \leq \lambda < \kappa$).
- $C(\alpha, \beta)$ is the closure of $\beta \cup \{0, I\}$ under the operations $+$, ϕ, $\lambda \mapsto \aleph_\lambda$, and $\langle \pi, \xi \rangle \mapsto \psi_\pi(\xi)$, where π is a regular cardinal $> \omega$ and $\xi < \alpha$.

We now list the proof-theoretic ordinal of several type theories, in ascending order.

Type Theory	**Ordinal**	**Proof**
ML_1	γ_1	[Acz77]
ML_n	γ_n	[Fef82]
ML	Γ_0	[Fef82], Aczel `citation?`
ML_1W	$\psi_{\aleph_1}(\aleph_{I+\omega})$	[Set93]
ML_nW	$\psi_{\aleph_1}(\aleph_{I_n+\omega})$	[Set98] `conjectured, not proved?`
MLW	$\psi_{\aleph_1}(\aleph_{I_\omega})$	[Set98] `conjectured, not proved?`

`It would be interesting to find out how` $\phi_{\Omega^\omega}(0)$ `and` $\psi_{\aleph_1}(\aleph_{I_\omega})$ `compare.`
`Check Setzer's latest papers`

References

[Acz77] Peter Aczel. The strength of martin-löf's intuitionistic type theory with one universe. In S. Miettinen and J. Väänänen, editors, *Proc. Symp. on Mathematical Logic (Oulo 1974)*, pages 1–32, 1977.

[AG02] Peter Aczel and Nicola Gambino. Collection principles in dependent type theory. In Zhaohui Luo, James McKinna, and Robert Pollack, editors, *Types for Proofs and Programs*, volume 2277 of *Lecture Notes in Computer Science*, pages 1–23. Springer, 2002.

[BP83] Paul Benacerraf and Hilary Putnam, editors. *Philosophy of Mathematics: Selected Readings*. Cambridge University Press, second edition edition, 1983.

[CH88] Thierry Coquand and Gérard Huet. The calculus of constructions. *Information and Computation*, 76:95–120, 1988.

[Dum63] Michael Dummett. The philosophical significance of Gödel's theorem. *Ratio*, 7:140–155, 1963.

[Fef64] S. Feferman. Systems of predicative analysis. *J. Symbolic Logic*, 29:1–30, 1964.

[Fef82] Solomon Feferman. Iterated inductive fixed-point theories: Application to hancock's conjecture. In G. Metakides, editor, *Patras Logic Symposium*, pages 171–196. North-Holland, 1982.

[Fef05] Solomon Feferman. Predicativity. In Stewart Shapiro, editor, *The Oxford Handbook of the Philosophy of Mathematics and Logic*, chapter 19, pages 590–624. Oxford University Press, 2005.

[G4̈4] Kurt Gödel. Russell's mathematical logic. In Paul A. Schilpp, editor, *The Philosophy of Bertrand Russell*, volume III of *The Library of Living Philosophers*, pages 125–153. Evanston & Chicago, Northwestern University, 1944. Reprinted in [BP83].

[Gir72] Jean-Yves Girard. *Interpretation fonctionelle et elimination des coupures de l'arithmetique d'ordre superieur*. These d'etat, Paris VII, 1972.

[Kle59] S. C. Kleene. Hierarchies of number-theoretic predicates. *Bull. Amer. Math. Soc.*, 61:193–213, 1959.

[KLN02] Fairouz Kamareddine, Twan Laan, and Rob Nederpelt. Types in logic and mathematics before 1940. *Bulletin of Symbolic Logic*, 8(2):185–245, June 2002.

[Kre60a] Georges Kreisel. La prédicativité. *Bull. Soc. math. France*, 88:371–391, 1960.

[Kre60b] Georges Kreisel. Ordinal logicas and the characterization of informal concepts of proof. In John Arthur Todd, editor, *Proceedings of the International Congress of Mathematicians (at Edinburgh, 1958)*, pages 289–299. AMS, 1960.

[Kre65] G. Kreisel. Foundations of intuitionistic logic. *Journal of Symbolic Logic*, 30(2):243–244, 1965.

[LM59] P. Lorenzen and J. Myhill. Constructive definition of certain analytic sets of numbers. *Journal of Symbolic Logic*, 24(1):37–49, March 1959.

[Lor55] Paul Lorenzen. *Einführung in die Operative Logik und Mathematik.* Springer, Berlin, 1955.

[Luo94] Zhaohui Luo. *Computation and Reasoning: A Type Theory for Computer Science.* Number 11 in International Series of Monographs on Computer Science. Oxford University Press, 1994.

[ML71] Per Martin-Löf. A theory of types. 1971.

[ML75] Per Martin-Löf. An intuitionistic theory of types: Predicative part. In H. E. Rose and J. C. Shepherdson, editors, *Logic Colloquium 1973*, pages 73–118. North-Holland, 1975.

[ML82] Per Martin-Löf. Constructive mathematics and computer programming. In *Logic, Methodology and Philosophy of Science, VI, 1979*, pages 153–175. North-Holland, 1982.

[Pal98] Erik Palmgren. On universes in type theory. In G. Sambin and J. Smith, editors, *Twenty-Five Years of Constructive Type Theory*, pages 191–204. Oxford University Press, 1998.

[Par92] Charles Parsons. The impredicativity of induction. In Michael Detlefsen, editor, *Proof, Logic and Formalization*, chapter 6, pages 139–161. Routledge, 1992.

[Poi06] Henri Poincaré. Les mathématiques et la logique III. *Revue de Métaphysique et de Morale*, 14(1):294–317, 1906.

[Rus06] Bertrand Russell. Les paradoxes de la logique. *Revue de Métaphysique et de Morale*, 14(5):627–650, 1906.

[Rus07] Bertrand Russell. On some difficulties in the theory of transfinite numbers and order types. *Proc. London Math. Soc.*, s2-4(1):29–53, 1907.

[Rus08] Bertrand Russell. Mathematical logic as based on the theory of types. *American Journal of Mathematics*, 30(3):222–262, July 1908.

[Sch65] Kurt Schütte. Predicative well-orderings. In J. Crossley and M. Dummett, editors, *Formal Systems and Recursive Functions: Proceedings of the Eigth Logic Colloquium Oxford, July 1963*, pages 279–302. North-Holland, 1965.

[Set93] A. Setzer. *Proof theoretical strength of Martin-Löf Type Theory with W-type and one universe.* PhD thesis, Universität München, available via http://www.cs.swan.ac.uk/∼csetzer, 1993.

[Set98] Anton Setzer. Well-ordering proofs for Martin-Löf's type theory with W-type and one universe. *Annals of Pure and Applied Logic*, 92:113–159, 1998.

[Spe55] C. Spector. Recursive well-orderings. *J. Symbolic Logic*, 20:151–163, 1955.

[Wan54] Hao Wang. The formalization of mathematics. *Journal of Symbolic Logic*, 19(4):241–266, 1954.

[Wan59] Hao Wang. Ordinal numbers and predicative set theory. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 5:216–239, 1959.

[Wea05] Nik Weaver. Predicativity beyond γ_0. 2005.

[Wey18] Hermann Weyl. *Das Kontinuum.* 1918. Translated as [Wey87].

[Wey87] Hermann Weyl. *The Continuum: A Critical Examination of the Foundation of Analysis.* Thomas Jefferson University Press, Kirksville, Missouri, 1987. Translated by Stephen Pollard and Thomas Bole.

[WR27] Alfred North Whitehead and Bertrand Russell. *Principia Mathematica.* Cambridge University Press, 1925–27. 3 vols.